MATHEMATICS CURRICULUM DEVELOPMENT FOR COMPUTERIZED ENVIRONMENTS: A DESIGNER-RESEARCHER-TEACHER-LEARNER-ACTIVITY[1]

Working Version

Rina Hershkowitz, Tommy Dreyfus, Dani Ben-Zvi, Alex Friedlander, Nurit Hadas, Tsippora Resnick, Michal Tabach
The Weizmann Institute of Science, Rehovot, Israel
Baruch Schwarz
The Hebrew University of Jerusalem, Israel

Introduction

The goal of this chapter is to shed light on the development of mathematics curricula integrating interactive computerized learning environments. Rather than describe and analyze one of the components in isolation from the others, we will try to give a comprehensive picture of the compound and long-term activity of curriculum development.

Curriculum development is the process of developing a coherent sequence of learning situations, together with appropriate materials, whose implementation has the potential to bring about intended change in learners' knowledge. The term 'knowledge' may be understood and interpreted quite differently by various parties such as decision makers, curriculum project team members, subject matter specialists, researchers, teachers, students and their parents. This is one reason why curriculum development may lead to tensions between some of these parties.

The situation is especially complex when the activity of curriculum development is aimed at learning mathematics in an environment in which the benefit from the potential of computerized tools has a central role. In their comprehensive chapter on "Computer-based learning environments in mathematics", Balacheff and Kaput (1996) explained why they think that the technology's power is primarily epistemological, and added:

> "While technology's impact on daily practice has yet to match expectations from two or three decades ago, its epistemological impact is deeper than expected." (p. 469)

In this chapter, we aim to show how such epistemological impact may lead to the design and realization of a curriculum, which does impact on the daily practices of teaching and learning mathematics in many classrooms. Although any curriculum development project is embedded in its own socio-cultural context, there are also many common features between them. In this chapter we describe and illustrate these common features. Rather than deriving them from theoretical deliberations, we will give meaning to them via specific examples of curriculum development. In other words, the specific examples will serve as appropriate windows through which curriculum development is seen as a comprehensive, theoretically and practically consistent activity. These windows belong to CompuMath, a large-scale curriculum development, implementation and research project for the junior high school level. The CompuMath project has been active for the past six years. It is the most recent cycle in a long-term process that was initiated more than twenty years ago and propagated in subsequent cycles of curricula. These cycles are all based on the same national syllabus, and

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in each of them our main goal was to design and create a learning environment in which students are engaged in meaningful mathematics.

By meaningful mathematics we mean that students' main concern are mathematical processes rather than ready-made algorithms. The following mathematical processes will serve as a representative sample:

- inductive explorations: generalizing numerical, geometrical and structural patterns, making predictions and hypotheses;
- explaining, justifying and proving these hypotheses.

These processes arise for the students in familiar problem situations as natural means for investigating and solving the problem, rather than as ritual procedures that are imposed by the teacher or the textbook.

In each cycle of curriculum development, we took into account the lessons learnt from research and development in previous cycles, theoretical frameworks and relevant cultural artifacts which were available at the time (for example computerized tools in the CompuMath cycle) and, above all, our socio-cultural view about mathematics and the learning of mathematics. Such a curriculum development cycle is a comprehensive process, which consists of three stages: The first stage involves *design* considerations, before starting the actual development and research work; the second consists of a *first design of the activities* and their implementation in a few classrooms, accompanied by classroom research on learning and teaching practices (observations, data collection and analysis); and the third stage comprises the *creation of coherent sequences of redesigned activities* forming a complete curriculum and its implementation, including the dissemination of the curricular aims and 'spirit' on a national scale.

When discussed theoretically, the potential of computerized learning environments refers to what *might happen* in such environments. "Developmental research" (Cobb, 1998) involving computerized learning environments tries to show an existence example of what *can happen*, and to serve as a window for investigating *how it happens*. Practice oriented educators are faced with the challenge *to make things happen for large populations of students and teachers*. In the CompuMath activity of curriculum development, we had all of these goals in mind at all times. The team thus has to deal with many faces of theory, research and practice of development and implementation, where practices are fed by theory and research and vice versa. The team functions as a huge cell eager to live and develop, but whose life is in large part determined by its interaction in an unknown world. Through this interaction the curriculum development activity constantly redefines its own components.

This chapter has two parts. In the first part, we describe the specific characteristics of the three stages in terms of various components of the curriculum development process, as listed below. Particular attention will be paid to the issues related to the use of computerized tools. In the second part of the chapter, we take a more longitudinal view and present three narratives that are representative of the process of curriculum development. While these narratives are based mainly on our own experience, they do relate strongly to theories and work done by others, and thus contribute to the provision of a comprehensive view of mathematics curriculum development for computerized environments. Each narrative focuses on a small number of major concerns in curriculum development including the role of research (the section on geometry), the choice and potential problems of computerized tools (the section on algebra), and project work and learning trajectories (The section on statistics).

The main issues in all three stages of curriculum development, fall into four groups. First, mathematical content and syllabi as given by external agents, explicitly or implicitly, as well as possible national standards and international trends. Second, the participants in the process, from project team members to students, teachers, classrooms, principals and other functionaries of the school system, each with possibly different roles at different stages. Third, the theoretical, socio-cultural and technological background of different participants, including their knowledge and experience of research; and fourth, the actual process of design, development and implementation. The sections of the chapter are organized around these issues.

Stage I: Pre-design Considerations

Syllabus, Curriculum and Standards

In the introduction to his chapter "Curriculum, Goals, Contents, Resources", Kilpatrick (1996) claimed:

> "In most of the countries of the world, the 20th century has witnessed a rather strong stability in the structure of the school and university mathematics curriculum even as waves of reform have swept across the surface." (p. 7)

We propose to refine this diagnosis by distinguishing between the terms *syllabus* and *curriculum*. In many countries, a central syllabus is prescribed by some authority. This syllabus is usually expressed as a list of contents and/or skills which students at a specific age or level should know. Often an external, central examination with a crucial role in the students' academic future is imposed, based on this syllabus. It seems that in the above claim about curriculum stability, Kilpatrick related to what we will call here, for purpose of clarity, a *syllabus*.

In contrast, a curriculum, as we understand it, is a far more comprehensive notion. Its goals are intended changes in learners' knowledge (in the widest possible sense), and it is expressed as a coherent sequence of learning situations, together with the necessary materials such as textbooks, teacher guides and many other components created in order to implement the intended changes. Hence, a successful curriculum mediates teaching and learning in actual classroom practice in such a way as to bring about intended change in learners' knowledge.

One of our goals in this chapter is to show that even in places where an official prescribed syllabus does exist, and even when this syllabus stagnates for a long time, far-reaching change is possible. Such change can be achieved through a carefully designed and developed curriculum by means of approaches to problem solving, classroom practices, the incorporation of socio-cultural tools like computers, etc. In this manner, genuine change in mathematics learning and teaching (official or not), and even in the mathematical content itself, can be achieved. An example is provided by the "Calculator-aware number curriculum" in England (Ruthven, 1999), where a government-supported curriculum development project had a strong influence on the design of a national curriculum (what we would call a syllabus) in mathematics.

Syllabus and curriculum (as products) may be seen as two poles between which the curriculum development activity is taking place. In many countries it is common that materials for classroom use are produced by a single writer, usually supported by a publisher. Such materials tend to reflect the bare syllabus, presumably because a lone writer lacks the resources and/or the motivation for change. In other cases, as we will demonstrate, the curriculum is immensely richer than the syllabus. In these cases, there exists no direct and easy translation of the syllabus into a curriculum that supports the intended changes in learning and teaching processes in the classroom. There are two crucial reasons for this; one is that the syllabus is a static list which deals only with the questions of *what* contents are to be learned, whereas the curriculum leads the practice of doing mathematics in the classroom, and as such it also deals with the *how*. The second reason is that unlike a syllabus, the curriculum relates to mathematical processes such as visual reasoning, hypothesizing, and investigating.

In order to bridge the gap between syllabus and curriculum, documents intended to inform and lead reform efforts have been published. The most impressive of these are the NCTM Standards (NCTM, 1989), published as the result of a five year effort by leading mathematics educators in the United States, and revised on the basis of a decade long follow-up effort (NCTM, 2000). The NCTM standards go far beyond the bare list of mathematical topics; on the other hand, they are still far from constituting a curriculum that can be implemented in classrooms.

If no appropriate guidelines such as the NCTM Standards are available to curriculum developers, they have to develop 'internal' standards, to guide their curriculum development work. Such internal standards may be implicit or explicit. In the context of the CompuMath project we dealt with the reality of a rather rigid official syllabus on the one hand, and long term government support for innovative curriculum development projects, on the other. In the absence of external standards we developed internal standards. Some of these were adopted or adapted from previous cycles of curriculum development, others were decided upon explicitly during stage 1, and still others which may have been implicitly influencing some of the work at stage 1, became explicit only during stage 2. The standards of the CompuMath team at stage 1 were:
1. Inquiry (observing, hypothesizing, generalizing, and checking) is a desirable mathematical activity.
2. Mathematical activity should be driven by the goals of understanding and convincing.
3. Proving is not only the central tool for providing evidence that a statement is true, but should also support understanding why it is true.
4. Mathematical activity should take place in situations that are meaningful for the students.
5. Mathematical activity must stem from previous knowledge (including intuitive knowledge).
6. Mathematical activity should be largely reflective.
7. Mathematical language (notation systems) fosters the consolidation of mathematical knowledge; it should be introduced to students when they feel the need for it.
8. Technical manipulation is not a goal in itself, but a means to do mathematics.
9. Computer tools support and foster the above and beyond.

Some of these standards were well formulated already at this stage, whereas others were not. For example, standards 1, 3 and 6 deal with the character of students' mathematical activity and with forms of their mathematical knowledge, topics with which the team had extensive experience from previous cycles; therefore we were able to give them a definitive formulation. On the other hand, standard 9 relates to computer tools which were rather new to us at the time; it is optimistic but vague, because at that time, so was our knowledge about the potential of computers as a regular and integral part of classroom activity. The development and elaboration of this standard in the course of the project will be discussed below.

<u>Participants in the Curriculum Development Activity</u>

At the pre-design stage, only a few people are actually participating in the activity of curriculum development, namely the members of the R&D team. For the CompuMath project, these were mostly members of the mathematics group at the Department of Science Teaching of the Weizmann Institute. The members of the team included designers who specialized in producing written materials, experienced teachers working with the team part time, and researchers in mathematics education. At that stage, the team functioned as "designers" of the future curriculum; they regularly imagined how a particular design would play out in classrooms with students and teachers, virtual participants in the activity of curriculum development, which the designers had in mind. One of the team members described this as follows:

> "I am thinking about the big mathematical questions to be addressed by means of a classroom activity; when doing this, I think about the children intuitively, based on my teaching experience. This is why I am much better at developing junior high school materials than, say elementary school materials: I have many years of teaching experience at the junior high school level. I am playing things through with virtual children – the kind I know from experience – and this often leads me to reconceptualize or reformulate, before a particular activity is even tried out in the classroom. Moreover, classroom trials of activities have often failed because teachers acted differently from what I had expected. Therefore, I now imagine a virtual teacher and often write a brief teacher manual for her before I ever send an activity to be tried in a classroom. This again has frequently led to changes in the design of planned activities before they were tried out."

<u>Tools, theories and research</u>

At this stage of the CompuMath project, we invested a considerable amount of time and effort in analyzing various computerized tools and establishing criteria for choosing the technology to be incorporated in our future work. We list here the main criteria that determined our choices, together with the underlying theoretical considerations, and explain in a general manner how the tools we chose actually satisfy the criteria. In the sections on geometry, algebra and statistics, this discussion will be completed by means of evidence for the potential of the chosen tools to support curricula that live up to our internal standards in specific content areas.

The primary consideration we used in choosing a piece of software for a specific mathematical topic, was the degree of adaptation of the software to the deductive nature and the content structure of the topic. This led us to define the following three more specific criteria;

1. The *generality* of the tool, its applicability in different content areas, its availability and its cultural status. Most tools have multiple uses. For example, a spreadsheet such as Excel may be used to store and analyze data, to create sequences of numbers from other sequences of numbers by manipulating general symbolic rules, and to represent numerical data graphically. Similarly, function graphers, as parts of more comprehensive software packages, are used in many domains in science and mathematics. More broadly speaking, we considered the cultural nature of the tool. Spreadsheet programs and graphers are ubiquitous in various places and domains. Such tools may be used again by students later, even in their professional life.
2. The potential of the tools to develop and support *mathematization* by students working on problem situations. This can take the form of amplification and reorganization (Pea,

1985; Dörfler, 1993) and of experiencing new 'mathematical realism' (Balacheff and Kaput, 1996). For example, the power of a grapher to smoothly transform a function from its algebraic to its graphical representation, and the availability of the corresponding numerical data directly from the graph (by 'walking on it'), make it possible to deal with problem situations involving complicated functions at an early stage of learning. Similarly, the capabilities of spreadsheets enable students to explore the meaning of trends in data, and to use different representations to exhibit these trends. They thus provide students with opportunities to relate data mathematically. And finally, the dragging mode in dynamic geometry environments provides the means to investigate geometrical features as invariants of a changing figure.

3. The third criterion is what we call *communicative power*, (or the semiotic mediation power) that is the power of the tool to support the development of mathematical language. This concerns the nature of the symbol system used by the tool, and its relation to the symbol system more commonly used in mathematics. The symbol systems of graphers and dynamic geometry programs are in one-to-one correspondence with the symbol systems of mathematics. The symbol system of *Excel*, on the other hand, is intermediate between the formal algebraic symbol system and an informal verbal notation system.

All three criteria are closely related to the multi-representational nature of the tools, the support they give to transformations between representations, and to the manipulation of mathematical objects (drawings, graphs, tables). They also imply that the curriculum developer pay attention to the fact that every representation admits many different representatives of the same mathematical object (such as many different graphs of the same function), and that students may choose to transform between these (Schwarz & Dreyfus, 1995). Efficient problem solving in mathematics depends on the flexible manipulation of objects in different representations and notation systems (e.g., the algebraic, graphical, and tabular representations in algebra, the pie chart or the frequency stick chart in statistics, geometrical drawings). Actions or operations one can undertake in each of the representations are different in nature when one is limited to paper, pencil and ruler. Kaput (1992) calls a notation system such as the algebraic representation an action notation system, because it allows calculations and transformations. In contrast, he calls notation systems such as the graphical and tabular representations, display notation systems, because the activity of the user is generally confined to interpretation. This theoretical distinction between action and display notation systems does not hold any more when one uses computer tools which provide, in addition to the representations themselves, the option of passage among representations and user based manipulations. In this case, all representations become action notation systems: it is possible to 'walk' on a graph, to stretch graphs (scaling), to change geometrical shapes by dragging, to rearrange a table, and so on.

The three criteria led us to decide on a type of tool for each of the main topics in the syllabus: spreadsheets (for statistics and algebra), graphers (for functions) and dynamic geometry. The selection of a particular piece of software within these types was based on various additional criteria including user-friendliness, didactic power (e. g., how many graphs can be shown simultaneously) and more mundane considerations such as affordability and availability. Detailed discussions of the nature and potential of the chosen tools appear below.

Design and Development Plans

At this stage, design and development existed only as plans for future work. Our attitude during this stage, and during much of the initial development in stage 2, is expressed well in a few lines by Balacheff and Kaput (1996):

> "In challenging most traditional assumptions about teaching and learning, technology forces us to think deeply about all aspects of our work, including the forms of the research that need to be undertaken to use it to best advantage. Clearly, our most important work lies ahead of us." (p.495*)*

Although we were experienced curriculum developers, we faced a new learning arena. This challenged us to rethink our day-to-day work while creating new learning environments. We reflected on integrating and translating into realizable plans the standards discussed above, given the computer tools, and given limitations imposed by the educational system. As in the case of the 'virtual student' and the 'virtual teacher', this 'virtual design' emerged, at least partially, from our established practices and shaped our new ones. The decisions made at this stage focused on:

a. The nature of mathematics – we decided to broaden the learned mathematical contexts.
b. The scope – we decided to create a curriculum for all the central topics in the Junior High School syllabus (grades 7, 8 and 9).
c. The ways in which the tools would be incorporated – we decided to base the teaching-learning process on the regular use of tools, rather than to use them sporadically only.
d. The order in which we would develop the different topics – we decided, for example, to start with functions, and to base the learning on the use of multirepresentational tools (graph plotters), because their potential seemed obvious and relatively easy to adopt. Moreover, the possibility to use a large variety of functions (including linear and quadratic that appear in the syllabus), was an opportunity to create rich problem situations.
e. The characteristics of teaching-learning processes – we decided to amplify processes, which were started in previous curriculum development cycles, such as investigations of open problem situations, in which groups of two to four students deal with a broad variety of mathematical phenomena.
f. We were developing a non-traditional curriculum for a large-scale population of teachers and students – we thus decided to educate and train teachers in the spirit of the project's goals from the beginning.
g. We were fully aware to the novelty of the curriculum development as well as of our limited experience with it – we thus decided that sound research was going to be an integral part of our work.

Stage II: Initial Design-Research-Design of Isolated Activities.

The goal of stage II was the first realization of the plans and pre-design considerations, elaborated in line with the internal standards agreed upon in the previous stage and the knowledge and beliefs of the participants. This first realization consisted of the design of mathematical activities, and the investigation of their impact in trial classes. Because so many new objects and actions of curriculum development had to be taken into consideration, and because the team members could grasp them only gradually, the activities designed in this stage were isolated rather than in sequence. The overall continuum served as a somewhat vaguely envisaged background against which the isolated activities were designed. This stage was characterized by the following features:

1. Dilemmas concerning the translation of the contents prescribed by the syllabus into first trial activities, which conform to the standards of the emerging curriculum (2.1).
2. The new community of participants in the curriculum development activity (2.2).

3. The development of isolated activities as a dialectic process of design-research-design cycles, the theoretical framework of the research, and what we can (not) learn from it (2.3)
4. First models of implementation with a view to the larger population (2.4).

2.1 Dilemmas on the Way from Syllabus to Curriculum

During the first steps of development, developers cannot avoid thinking and rethinking the basic approaches to the specific mathematical content. Very often, the dilemmas arise from conflicts between the content as it appears in the syllabus and the approaches that were adopted in order to make full use of computerized tools. Such dilemmas serve as catalysts for rethinking approaches and methods, and for innovative solutions in the curriculum development work. We demonstrate this issue, by relating to three content areas.

- *Euclidean geometry* is one of the main topics in the prescribed syllabus, according to which it is to be taught in the second half of the 8th grade and during the whole 9th grade. Within Euclidean geometry, proof and proving are central. In the section on geometry, we discuss how familiarity with dynamic geometry software, and the awareness of its potential, put in question the role of proving, since invariant properties of geometrical figures can be observed visually, and justified inductively. This caused us to rethink the various roles of proof. As a consequence, we started to develop activities where proof is considered as a *tool for explanation,* by means of which students may understand and explain *why* the conjectured invariant attribute, which they discovered in the dynamic geometry environment, is really invariant. Similarly, we developed activities in which the crucial point cannot be discovered by dynamic geometry, for example because it is an impossibility; in such a case, proof is the only way to be certain about the conclusion. Examples for each of these types of activity will be given in the section on geometry.
- Within the topic of *functions*, the possibility of obtaining the graph of any function from its symbolic representation, to 'walk' on the graph, and to read from the graph the coordinates of special points like extrema, considerably enrich the teaching-learning of functions in junior high school. But, this power puts in doubt what is commonly presented as a main motive for learning calculus in high school: The ability to find the main features of a given function's graph. The dilemma arises whether to reduce the teaching of derivatives at high school, or to give it new motives, such as modeling by differential equations.
- Before starting the design and development of *algebra* activities, we had to make a decision about the approach. Several considerations led to the selection of a functional approach (Yerushalmy, 1997). First, the main focus of beginning algebra in the 7th grade is the generation of symbolic generalizations of number patterns, which can very naturally be seen as the discovery of the symbolic rule of a function. In addition, the use of a spreadsheet emphasizes the transition between dynamically varying numbers and their symbolic rules. Moreover, graphs can be produced when wanted. An example of the rich activities developed for beginning algebra is given in the section on algebra. The functional approach and the use of graphing software appear to create an appropriate opportunity to broaden the concept of solving equations. Solving equations was presented as finding the intersection points of the graphs of two functions, linear or not. Thus the graphical solution of a given situation became more fundamental than the algorithmic-symbolic one. On the other hand, students were required by the official syllabus to master the algorithm for solving linear equations, and we needed to develop suitable activities for teaching it. One of these activities dealt with the transformation of a given equation into an equivalent one. Students in the trial classrooms, who were

already familiar with the intersection point view of a solution, were asked to conjecture the graphical representation of an equivalent equation. Students, and even some of the teachers, were quite surprised to discover, by means of a computerized tool, that the equivalent equation is represented by functions that have a *different* intersection point (with the same x-coordinate). Thus we discovered that the functional approach does not well support the algorithm for solving a linear equation, and presented this topic differently.

Through these and similar cases we learned that we needed to use different approaches and points of view flexibly in curriculum development just as in problem solving.

2.2 Participants in the Curriculum Development Activity

The curriculum development team members continued, of course, to form the core, but additional participants were added at this stage: teachers and students in trial classes, which were not virtual any more. The team members as a group were involved with a large and complex array of interrelated tasks: design and development of activities, in-depth learning of the mediating potential of the computerized tools, and teaching in trial classrooms, including observation, investigation and analysis of teaching – learning processes. The role of several members of the team expanded considerably during this stage. For example, one central member of the team, an experienced teacher as well as developer, became part of the research team that investigated the trial teaching of activities on function in her own classroom (Resnick, Schwarz, & Hershkowitz, 1995). During this stage, many team members went through intensive processes of introspection and reflection on their own products and actions. This issue will be discussed in more detail in the next subsection.

On the other hand, teachers and students in trial classes began to have an impact on the development process and were thus integrated into the ‘community of participants’. The common denominator among all these new participants is that they were highly motivated to realize the goals of the project and aware of their potential impact on the curriculum. For example, in our ‘lab school’, the teachers (some of whom are members of the core team) used to meet regularly to create activities together, produce worksheets, and try them with their students. The students were aware that their role was important in evaluating the new approaches and activities. It happened very often that teachers, and even students, suggested to modify activities. At this stage, the researchers in the team videotaped some classroom activities. The students were generally willing to be videotaped, because they felt that they were part of an adventure and that the difficulties they encountered were precious data that served to improve the curriculum.

2.3 The Dialectic Process of Isolated Activity Development through Research

The core of the development work at this stage consisted of developing isolated activities in design-research-redesign cycles. The process was thus a dialectic one, during which design and research influenced each other.

The various dimensions considered in the design of each activity include the content and the overall mathematical approaches, the intended mathematical thinking processes (generalizing, hypothesizing, reflecting and justifying), the potential of the tool, the classroom organization (including redistribution of learning responsibilities between students and teacher), practices and socio-mathematical norms.

In trials of these early isolated activities, we were carried away by the exciting and surprising processes we observed, and by the extent to which they differed from what we had observed during the previous two decades of development and research. What started as naïve observation and documentation by taking field notes, was soon transformed into coherent research with videotape documentation and detailed analysis and interpretation. The need to describe, understand, explain and analyze what was going on in these classrooms naturally brought us closer to the concerns of socio-cultural psychology. Like many others (c.f. Perret-Clermont, 1993; Yackel & Cobb, 1996), we felt the shortcomings of cognitive theories, methodologies and tools we had at our disposal to describe and interpret learning and teaching processes in the classroom. We adopted activity theory (Kuutti, 1996) as the theoretical frame for the interactionist approach. The construction of knowledge was analyzed while students were investigating problem situations in different contexts. Research became a crucial component in the curriculum development activity.

Two types of research were interwoven in these design–research–redesign cycles. Both types might be called developmental research (Cobb, 1998), in the sense that they involve instructional development with research. The first used interviews with pairs of students, interlaced with development cycles. In the section on geometry we present an example which comprises several research–redesign cycles.

The second type is classroom research, which focused on investigating the ways in which the goals and standards of the intended curriculum were implemented. We illustrate this classroom research by means of an example in which we investigated students' processes of hypothesizing and reflecting, as well as the role of teacher in the orchestration of these processes. The example deals with the design-research–redesign cycles of an activity called 'Overseas' at the end of the year-long functions course in grade 9.

We had been working in a particular trial classroom from the beginning of the school year. The teacher was a member of the CompuMath team, and the main designer of the functions' activities. As team member, she was eager to try the new activities in her classroom. But as teacher, she also had to follow the official syllabus of the grade 9 functions course with her class. Hence the activities, in spite of being isolated and innovative, formed an integral part of the official syllabus.

Two team members observed each new activity in this classroom throughout the year. In this way, we accumulated experience concerning the development of learning opportunities through the power of the computerized tool, and inquiry during problem solving processes. At the same time, changes in classroom practices were noted. The observations, which were at first unstructured, became focused in the course of the year. The analysis of the observations and the conclusions we were able to draw, served as the basis for the design of other activities, as well as for improving the observed activities themselves at the next stage of curriculum development (Stage III).

Hence, in 'Overseas' we already inserted all the knowledge we had gathered from the development-research experiences in previous activities. The classes were carefully planned as a research arena with precise documentation. Specifically, we observed and videotaped a group of four girls during group work and collected all their products; we also collected the written products of the other groups and carefully recorded all whole class discussions. On the whole, we had a twofold goal: to develop and structure an "ideal" activity, and at the same time to examine and check its realization in the classroom. In particular, we examined students' ability in making hypotheses, their awareness of the quality of hypothesizing processes and the nature of different reflective processes in different phases of the activity. In addition, we investigated the orchestrating role of the teacher during the activity.

The research has been reported in detail in Hershkowitz and Schwarz (1999). Some of its conclusions are:

1. The power of the tool to deal with a large variety of functions makes rich problem situations possible; for example, in 'Overseas' students found a local maximum of a function which is a composition of a rational and quadratic function – a type of task not accessible to ninth graders without a graphical tool.
2. In rich problem situations, inquiry is a natural process. Students have and use the opportunity to move among representations in order to progress.
3. Asking students to make hypotheses about possible solutions *before* solving the problem is a valuable didactic technique. The students were able to delay the actual solution, and accept hypothesizing as a valuable activity.
4. Reflection, does not usually occur spontaneously but has to be initiated, for example by requiring students to write a group report on their inquiry process.
5. A teacher led synthesis in a session with the entire class is useful for many reasons. Students can be given an opportunity to report on their work and to practice participation in classroom debates, in which they can give, as well as obtain, critique. The teacher can use their reports to raise criticism and evaluation, as well as for a synthesis of the main processes students went through. The session thus affords another opportunity for reflection. Last, but not least, such a synthesis allows the teacher to define the common knowledge, which she expects the students to have gained.
6. The teacher's role during the synthesis session is crucial. In 'Overseas', for example, she made it clear that the goal is not to present results but to reflect on the process they had gone through, in particular how they had hypothesized possible solutions. Thus she conveyed that hypothesizing prior to solution is a socio-mathematical norm for her classroom.
7. It is advantageous to let students carry out the inquiry and reporting on the inquiry in groups, because social interaction in the group supports mathematical argumentation: students complete, oppose and criticize others' proposals, progressing towards agreement among the group.

Classroom research thus gave us a large amount of input in an area with which we had little experience from prior cycles of curriculum development: How to design extended activities based on rich problem situations into multiple phases, including inquiry by groups of students, reporting on the inquiry, and teacher led synthesis. We note in passing that a more detailed model for an activity contains, in addition, preparatory individual homework before the group inquiry, and summary individual work after the synthesis. We also learned about the teacher's role in inviting students to act differently in each of the phases, so as to have different opportunities for reflection.

During this stage we accumulated several activities in each topic, which incorporated the research results about rich problem situations, activity design, use of computer tools and teachers' and students' roles in the learning process. These model activities later served as milestones for the further development (see Heid, Sheets & Matras, 1990, for parallel experiences).

2.4 First Models of Implementation

Implementation takes place at different stages of curriculum development activity. Teachers who chose to teach with the project materials needed a lot of support both before and during their teaching, since every component is radically new - the technological tools, the organization of the learning environment and learning processes, the kind of open-ended problem situations and their multiphase structure, the kinds of learning, the methods of teaching and the ways of evaluating students.

When a sufficient number of activities on functions had accumulated, we organized an in-service course (about 60 hours, during the summer vacation). Teachers from various schools went through the same learning practices, in the same learning environment, as students in the trial classes. They then reflected on each activity as students and as teachers. About half of them volunteered to use these activities and others that would be developed during the following academic year. They received intensive support, mainly in bi-monthly meetings with team members. They received the new materials, and discussed ways of teaching and the rationale and practice of evaluating students learning in the new environment. An additional, quite different role of this group of teachers was to provide feedback from their classes. This feedback was invaluable in the next stage of curriculum development, as well as in the establishment of curriculum implementation practices.

At the same time, a different model of implementation was used; a long-term implementation model, which we call the ‘fan model’, since it propagates from lead teachers to other teachers. This model formed part of a larger project concerned with the use of computers in teaching and learning the mother tongue, foreign languages and mathematics in all elementary and junior high schools of a particular mid-size town.

Two or three teachers from each school were chosen as leaders to introduce computers for teaching mathematics in their school. They received personal computers, as well as instruction in basic literacy in the most common uses of computerized tools; they were also connected via an electronic network and met at a municipal teacher center one day per week. Within this framework, they took a two-year course on mathematics teaching with computerized tools. Although we had no influence on the choice of the participating teachers, some established very creative ways of teaching. For example, some teachers asked their students to work in pairs and to create their own investigation project. They encouraged the students to reflect on choosing the subject, the ways of posing questions, the openness of the project, and so on. In some classes, each pair of students was asked to give their project to another pair for criticism, after evaluation criteria had been established in whole class discussions. As a result of our involvement in this project, all junior high school classes in this municipality now use the CompuMath curriculum.

Already during the second stage, we thus had opportunities to learn about the potential of different implementation models and different types of in-service courses. We came to know some of the difficulties and limitations in leading reforms in general, and reforms involving computerized tools in particular, in a large scale population. This issue will be taken up in detail in the next section.

Stage III: Expansion

Having an impact on a large number of students and teachers is the raison d’être of curriculum development projects in general and of CompuMath in particular.

The metaphor of throwing a stone into still water is quite appropriate here. The stone causes waves to propagate outward in circles. These waves are forceful near the place where the stone fell, but become progressively smaller with distance. If only a single stone is used, the waves also fade out with time. In order to preserve the intensity of the waves, one should continue throwing stones into the water. In CompuMath, we found that the best way to do this, is to offer the teacher a large and varied collection of activities, which combine together to cover a whole continuum with a considerable breadth of choice. The breadth allows each teacher to choose from the already developed materials, her own trajectory along the continuum in a way that is appropriate for her class. We were thus led to expand the project along three main lines. The first is the ‘democratization’ of the teacher and student population using the curriculum: Teachers and students in classes unknown to the project team, with their own motives, drives, needs and school conditions. These may contain elements which are quite orthogonal to the approach and underlying philosophy of CompuMath, as other agents throw other stones, the waves of which interfere with those of the CompuMath project. The second line of expansion is the design and development work, which turned the project materials from a collection of isolated activities in stage II into a broad and flexible continuum. The third is research, of a more global nature, which emerged as an integral part of stage III. These three lines of expansion will be discussed in the following three subsections.

Expansion of the population

The social texture of the participating teacher and student populations in the third stage was radically different from the previous ones. The expansion to a wider population implied a large measure of 'democratization' for both learners and teachers. Regarding *learners*, it consisted in the adaptation of the 'intended curriculum' to a wider spectrum of possible learners. Some information on possible extensions and needed adaptations was gleaned from the experience during the second stage. For example, the students who participated in the 'Overseas' activity belonged to the upper 60% ability level of the general population. During this activity, groups that finished the inquiry phase early were asked by the teacher to prepare a worksheet about 'Overseas' for students whose level is a bit lower. In the whole class session at the end of the activity, one student presented her worksheet. It was a sequence of coaching questions such as 'draw the surface area graph using the graphical calculator', 'walk on it' and 'read off the minimal point'. The prescriptive character of this worksheet contrasts with the openness of the original activity. This worksheet presentation triggered a class discussion about learning modes suggested by activities like 'Overseas'. Some students, even though they were able to solve the open version of the activity, preferred a more closed version, or felt that students in other (lower level) classes would prefer it. Such a preference reflects a different learning norm. Extending learning situations to a wider spectrum of learners should and did take the existence of such norms, as well as the abilities and motivation of students in the lower 40% of the population into consideration.

With the expansion of the project waves to the heterogeneous general population of learners, to an anonymous population of teachers, principals, inspectors and even parents, less support and monitoring is flowing from the team to each classroom, and less information is coming back from classrooms to the team than in stage II. The teachers in these classrooms have varying degrees of commitment, and as a consequence the degree of implementation of the project varies from sporadic activities, to the adoption of the entire approach and set of project materials. The team initiates new ways to encourage and support innovations in schools, without impinging on the schools' autonomy.

From isolated activities to a continuum

The second line of expansion in stage III is the creation of chains of activities that have the potential to lead to long-lasting cognitive gains. These chains combine and shape the isolated activities developed in Stage II into a whole. They were elaborated according to the following perspectives:

- The structure of the content to be learned.
- The standards of the project (see Stage I).
- The power as well as the limitations of the computerized tools (see the section on algebra).
- The available time for computer use in school.

- The lessons drawn from the design-research cycles in the pilot classes in stage II.
- Activities should offer opportunities to capitalize on lessons from previous activities.

A typical chain of activities in the functions course, for example, constitutes a unit of six to eight lessons, which includes a key activity, a consolidation activity, several additional activities, and several follow-up corners, each with its purpose to be described below.

A unit starts with a *multi-phase activity* based on an *open problem situation* in the computer lab. This activity was typically designed and investigated in stage II. In stage III, it was revised in the light of lessons we had learned from research and observations in stage II, as well as its role as the key activity of the whole unit. One of the main goals of this opening activity is to create opportunities for students to *deal informally, from the start with all concepts, relations, and representations to be learned in the unit* as a whole. During the inquiry phase, tasks are set that allow these concepts to arise, without requiring a full treatment. We have thus chosen a holistic, frontal, and bold approach to learning rather than a linear one.

Immediately after the key activity, students are presented with a *follow-up* activity, which elaborates, formalizes and consolidates the informal knowledge constructed in the key activity. In other words, the teacher and the follow-up activity function as agents to facilitate a process of mathematization. The follow-up activity may contain some new elements, such as an efficient strategy, a new algebraic technique, or more formal language for one of the concepts. Very often observations in stage II are the source of such elements. For example, ‘interesting mistakes’, such as wrong verbal or graphical hypotheses, incomplete strategies, inefficient representation, are presented as possible solutions to be accepted or refuted. The intention is to create opportunities for students to reflect on these ‘mistakes’, through dialectical processes. We hope that teachers will pick up these ideas, accumulate interesting responses from their own classroom, and invite their students to reflect on them.

Additional activities are typically different problem situations, each giving rise to a multi-phase activity, with or without a computerized tool. The teacher is may select two or three for her class, or even replace the opening activity by one of them. These activities are not intended to require basically new content or processes. One of their goals is to lead students to more autonomy; there are fewer instructions (such as to use a certain representation) than in the opening activity.

We have observed that during the inquiry phase, especially when students are working with computerized tools, they have strongly varying strategies and rhythms. Two additional corners serve to keep the students “together” The *‘beacon’* section provides clues and support in various places in each of the open problem situations (the opening one as well as the additional ones), for students that may need such support. The ‘*see if you can’* corner offers challenging questions on the problem situations for students who finish the inquiry earlier than others.

A typical unit also contains a collection of *homework tasks* (without the computerized tool), a summary, usually in verse, in the *poet’s corner,* which summarizes the most important issues in the unit, and a ‘*keep fit’* corner of short tasks to practice algebraic techniques.

Finally, two further corners are devoted to interactive reading of mathematical texts connected to the unit, and to reflection on actions, content and learning processes. Throughout the unit reflection is encouraged, either as a monitoring action during a problem solving process or soon after it. In the *‘looking back’* corner students have opportunities to reflect globally on the whole unit.

In summary, each unit is organized so as to generate a dialectic process in which students deal with the key concepts and processes of the unit in different contexts, from different angles and with different purposes. Such a dialectic process fits the cognitive accumulation of knowledge well. It emerges from the key activity, in which the concepts and contents are linked mainly through the context of the problem situation, rather than through the mathematical structure. Each of the following tasks in the unit deals with a specific aspect, such as the formalization of the mathematical knowledge or links between different elements; moreover, quite a few of the tasks at the same time stress different aspects based on the standards, for example the students' mathematical autonomy, or the development of their reflective ability.

Research

When the isolated activities were redeveloped and integrated into a whole curriculum, new research questions, such as the construction of the knowledge of individuals over an extended period of time, became very important (Hershkowitz, 1999). The construction of knowledge via learning trajectories of students in the space of the classroom, can be examined by tracing individuals as well as groups within and across activities. For example, the need arose to investigate the ways in which hypothesizing processes develop in the first few months of algebra. The researcher created a 'diary' for each student and for every task from the beginning of the year. In this way she expects to be able to achieve three things in parallel: trace for each individual student how the notion of conjecture is constructed, investigate how conjecturing processes develop in different types of students, and follow the entire class through an activity in order to find out about the conjecturing potential of the activity, and to improve it if necessary. From this combination of observing individuals and the entire class, we expect to learn about the continuum, and be able to adapt it better to fostering the ability of conjecturing.

Another researcher focused on the progress over time of the relationship between shared knowledge and individual knowledge construction. She has begun observing and documenting the common, as well as the separate work, of a pair of students in activities scattered along the entire 7th grade algebra course. She is attempting to understand and interpret each student's construction of knowledge, as well as the social interaction between the pair. The intention is to focus on relationships between the changes in individual knowledge and the changes in shared knowledge, as well as changes in the way the students interact. The problem of individual versus shared knowledge is central to our curriculum. Many questions arise, such as to what extent the composition of student groups can influence the type and effectiveness of the interaction and the ensuing learning. Such research has feedback on stage III development. Interaction in a group and with the entire class, as well as with the computer tool, have a strong influence on building the continuum, since curriculum development includes also the planning of the type of collaboration within each phase of a multiphase activity.

Finally, some of our research focuses on outcomes, in particular on the effect of the curriculum on knowledge structures. In one such study we characterized students' function concept images at the end of the CompuMath course on functions (Schwarz & Hershkowitz, 1999). Specifically, this research does not show how to do curriculum development but rather what is its effect. This was done partly to satisfy our curiosity, and partly to provide data to principals, inspectors and other interested parties.

This concludes the first part of the chapter, in which we have described the characteristics of the three stages of curriculum development. In the second part, we will consider selected aspects of the development process for the various mathematical topics. For each topic, we will focus on a few issues particularly salient for this topic. Since many of the illustrations in the above description of the three stages were chosen from the topic of functions, we present below narratives from the three remaining topics: geometry, algebra and statistics.

Geometry: Concepts and Justifications

General questions such as "why teach geometry?" or "what should the study of geometry entail?" have been treated extensively in Lehrer & Chazan (1996). In this section we will focus on two main issues in learning geometry: the construction of concepts and the role of proof and proving. We concentrate on these issues, because they appear in the syllabus in many countries and are decisive for geometry learning in many classrooms.

When learning geometry, students are usually presented with a geometrical figure – the mathematical object – represented by a drawing. The geometrical figure is the concept itself, and the drawing is a representative of the geometrical concept(s). Laborde (1993) expressed this as follows: "*drawing refers to the material entity, while figure refers to a theoretical object*"(p. 49). She made it clear that there is always a gap between drawing and figure for the following reasons.

a. Some properties of the drawing are irrelevant. For example, if a rhombus has been drawn as an instance of a parallelogram, then the equality of the sides is irrelevant.
b. The elements of the figure have a variability which is absent in the drawing. For example, a parallelogram has many drawings, some of them are squares, some of them are rhombuses, and some of them are rectangles.
c. A single drawing may represent different figures (Yerushalmy and Chazan, 1990). For example, a drawing of a square might represent a square, a rectangle, a rhombus, a kite, a parallelogram or a quadrilateral.

When a single drawing is used as representative of a figure, these three differences between figure and drawing may turn into three connected obstacles to learning.

The isolated drawing is thus ambiguous as a representative of a figure, especially when students are engaging in geometrical situations using pencil and paper. A proper representation of a figure should include an infinite set of possible drawings. *Dynamic Geometry tools* (e. g., the dynamic version of the Geometric Supposer, Cabri géomètre, the Geometric Sketchpad, and the Geometry Inventor) offer this feature. They allow the student to 'drag' elements of a drawing, and thus enable the production of an infinite set of drawings for the same figure, all of which have the generic attributes of the figure (thus overcoming the problem raised by b). This 'variable' method of displaying a geometrical entity solves the problem c) of the ambiguity of a unique drawing as a representative of the entity. Moreover, it stresses the intrinsic attributes, the invariants of the entity, thus going a long way toward overcoming a). It is for these reasons, that we chose to adopt a dynamic geometry environment as computerized tool for the geometry component of CompuMath. We decided to use the Geometry Inventor (1994), mainly for reasons of availability; an additional reason was the ease with which the geometric variation of the elements of a figure could be represented in a Cartesian graph.

Research indicates that students engaged in dynamic geometry tasks are able to capitalize on the ambiguity of figures in the learning of geometrical concepts (Hoyles & Jones, 1998). In addition, Goldenberg & Cuoco (1998) claimed that:

> "Dynamic geometry offers an interesting arena within which to watch students construct or reconstruct definitions of categories of geometry objects, because it allows the students to transgress their own tacit category boundaries without intending to do so, creating a

> kind of disequilibrium, which they must somehow resolve. Confusions can be beneficial or destructive. Understanding how students resolve such conflicts will help us devise educationally better uses of the software, ones that maximize the opportunities and minimize the risks of the confusions created by such transgression of accepted definitions." (p. 357)

The above discussion is relevant not only to the choice of a particular technological tool but also to the two main issues to be discussed in this section, the construction of concepts and the role of proving. In each case, we will start with a short presentation of research that had an impact on our work, and then demonstrate how research and development interweave in the development of an 'isolated activity'.

<u>Concept construction</u>

A considerable part of the research on concept construction in geometry focused on the role of different examples in the construction of concept images. Among the examples of a concept one can often identify a 'prototype', which is a popular example typically drawn by many students when they are asked to provide a representative of the concept. For example, Hershkowitz (1987) found the following prototypical behavior concerning altitude in a triangle: Students usually draw the altitude inside the triangle, even the altitude to a leg of an obtuse angle. More than that, during the process of learning the altitude concept in a traditional setting, more students constructed the prototypical concept image than a flexible (correct) one. Yerushalmy & Chazan (1990) also reported that teachers and students were unwilling to draw exterior altitudes for obtuse triangles.

Dynamic Geometry software enables the design of activities in which students investigate the relevant properties of the figure by means of dragging. Such activities can support students in constructing a more appropriate concept image. More specifically, such an activity can lead students to construct an altitude in a acute triangle and then drag the vertices in such a way that they see the altitude move outside the triangle. During Stage II of the CompuMath project, such an activity was designed. The team member who proposed and elaborated this activity described the process as follows:

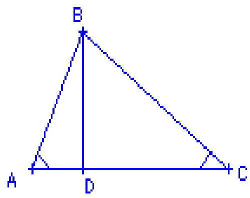
> "I knew that students were likely to develop a limited concept image of altitude I needed an activity that would work on this point. So first, I sat down at the computer and played around with triangles and their altitudes. Obviously you can drag the altitude outside the triangle, but simply dragging will not achieve much; before dynamic geometry students also saw obtuse triangles with altitudes outside. I needed a task that would engage them. So I imagined some [virtual] students; I know from my teaching experience how they might think, and what would catch their attention. Suddenly, I noticed something I hadn't explicitly thought about before: It never happened that a single altitude was outside the triangle. This was what I needed: The students would try to generate a triangle with one altitude outside; they would be surprised at their inability to find one, but not immediately understand why. On the other hand, the reason is clearly accessible to them and more importantly, it is directly connected to my main aim, namely to help them establish the connection between the sides of the obtuse angle and the altitudes to these sides, which are always outside the triangle. The option to combine the dragging mode of the object itself, with the surprise which raises a need for explanation, may influence the creation of more accurate concept images."

Dragging the vertices in task 1 enables students to see many different cases, including some with outside altitudes, or altitudes which coincide with one of the sides of the triangle.

The Altitude Activity

Task 1. Draw a triangle with an altitude to the side AC. Investigate the connection between the position of the altitude and the two angles <A and <C.

Consider three cases: the altitude is inside the triangle, outside the triangle, coincides with one of the sides.

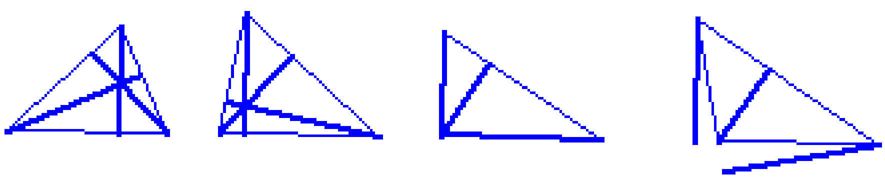


Explain the connection between the position of the altitude and the sizes of the angles <A and <C.

Task 2. Construct the two other altitudes and predict, without dragging the triangle, whether each of the following results, is possible. Then check, and explain:
- Two altitudes are inside the triangle, and one is outside.
- Two altitudes are outside the triangle, and one is inside.
- All the three altitudes are outside the triangle.

By describing the connections between the position of the altitude and a relevant attribute of the triangle's angles, students may create visual and verbal connections. Such an activity may help students to generate empirical evidence in order to make transition from the particular to the general case and from empirical to analytical reasoning, and thus to overcome the tendency to construct a limited concept image.

In task 2, students make use of the connections they have established, to investigate more complex situations in which all three altitudes are involved. While manipulating the triangle on the screen, trying to observe the relationships and to explain them, students have opportunities to visualize many representatives of the figure and connect them to the triangle's attributes; thus they have opportunities to construct a more sophisticated concept image. By dragging, students realize that in an obtuse triangle, two altitudes are drawn to the legs of the obtuse angle, and therefore both these altitudes fall outside the triangle (see the following figure).



It is worth noting that in a preliminary questionnaire to this activity, a group of students was asked to draw altitudes to a marked side of triangles drawn on paper. Their performance was similar to those described in Hershkowitz (1987). After the activity students answered the same questionnaire with few or no mistakes.

The altitude task is an example of what Laborde (1999) has characterized as new kinds of tasks which arose out of yearlong evolution of development work in dynamic geometry environments:

- "tasks in which the environment allows efficient strategies which are not possible to perform in a paper and pencil environment;"
- "tasks raised by the computer context; i. e. tasks which can be carried out only in the computer environment." (p. 306)

As she pointed out, such tasks may create "intriguing visual phenomena which are not expected by students. The only way of explaining those phenomena is recourse to theory" (p. 300). This is exactly the intention of the altitude task. Experiencing the impossibility of one outside altitude, while dragging the vertices of the triangle with its altitudes on the computer screen, creates surprise and leads students to a dilemma: Should they continue the empirical search (for a triangle with one altitude outside), or should they attempt to understand the impossibility? This dilemma lets students experience a mathematical need for proof, as will be described in the next subsection.

<u>The Role of Proof in Dynamic Geometry Environment</u>

For generations proofs were considered as tools for verifying mathematical statements and showing their universality. Hanna (1990) mentioned Leibniz who believed that

> "a mathematical proof is a universal symbolic script which allows one to distinguish clearly between fact and fiction, truth and falsity" (p. 6).

Thus the two classical roles of teaching proofs are to teach deductive reasoning as part of human culture and to verify the universality of geometric statements. Experimenting, visualizing, measuring, inductive reasoning and checking examples were not counted for this purpose.

Recently there has been a change in this approach for several reasons.

(i) The failure of teaching proving tasks in school

The teaching of mathematical proof appears to be a failure in almost all countries (Balacheff, 1988). Only 30% of the students in full year geometry courses that teach proofs (in the U.S.A.) reach a 75% mastery in proving (Senk, 1985). Every teacher, who teaches a traditional geometry course, can confirm these findings.

Moreover, students rarely see the point of proving. Balacheff (1991) claims that if students do not engage in proving processes, it is not so much because they are not able to do so, but rather that they do not see any reason for it (p. 180). High-school students, even in advanced mathematics and science classes, don't realize that a formal proof confers universal validity to a statement. A large percentage of students states that checking more examples is desirable (Fischbein & Kedem, 1982; Vinner, 1983). Many do not distinguish between evidence and deductive proof as a way of knowing that a geometrical statement is true (Chazan, 1993). After a full course of deductive geometry, most students don't see the point of using deductive reasoning in geometric constructions, and remain still naive empiricists whose approach to constructions is an empirical guess-and-test loop (Schoenfeld, 1986). They produce proofs because the teacher demands it, not because they recognize it as necessary in their practice (Balacheff, 1988).

(ii) The role of proof, and the goals of teaching proofs

For mathematicians, proofs play an essential role in establishing the *validity* of a statement and in *enlightening* its meaning. An analysis of teaching materials indicates, that the social and practical importance of proofs in mathematical activities remains hidden, and it is important to create classroom activities in which the student becomes aware of that aspect of proofs (Balacheff, 1988, 1991).

Hanna (1990) distinguishes between two kinds of proof: (a) proofs that show only that the theorem is true, by providing evidential reasons; (b) proofs that explain *why* the theorem is true, by providing a set of reasons that derive from the phenomenon itself. Hanna (1995) asserts that the main function of proof in the classroom is to promote understanding. Similarly, Hersh (1993) believes that in mathematical research the purpose of proof is to convince, but in the classroom the purpose of proof is to explain.

(iii) The existence of Dynamic Geometry environments

The advent of dynamic geometry environments raised a question about the place of proof in the curriculum, since conviction can be obtained quickly and relatively easily: The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence to convince themselves of its truth (De Villiers, 1998). The role of proof is then to provide the means to state the conjecture as a theorem (Yerushalmy & Houde, 1986), to explain *why* it is true, and to enable further generalizations.

Researchers have investigated how students function in open inquiry activities in computerized learning environments that support experimentation, conjecturing, checking invariant properties of a figure, and thus lead to conviction (Yerushalmy, Gordon & Chazan, 1993). Yerushalmy and Chazan (1987) pointed to the importance of problem posing in the design of activities in geometry. De Villiers (1997, 1998) illustrated how one can enrich investigations in dynamic geometry environments by asking 'what if' questions, and use them to make generalizations and discoveries. He claimed that in this case the search for proof is an intellectual challenge, aimed at understanding *why* the conclusion is true, not an epistemological exercise in trying to establish truth. In addition Goldenberg, Cuoco and Mark (1998) stated that:

> "a proof, especially for beginners, might need to be motivated by the uncertainties that remain without the proof, or by a need for an explanation of why a phenomenon occurs. Proof of the too obvious would likely feel ritualistic and empty" (p. 6).

Dreyfus & Hadas (1996) argue that students' appreciation of the roles of proof can be achieved by activities in which the empirical investigations lead to unexpected, surprising situations. Activities of this kind let students experience the need for proof in order to explain the surprising findings, and sometimes even in order to be convinced what are the correct conclusions. Different kinds of activities were developed in the CompuMath project in this "spirit".

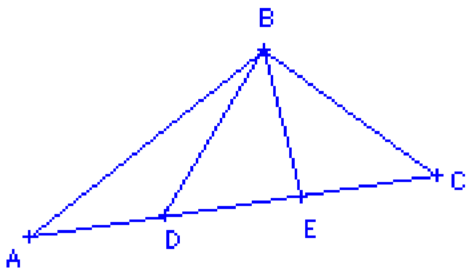
a. Comprehensive inquiry activities, where the geometrical fact discovered as invariant of a geometrical feature is surprising. This surprise is the trigger for the question *why* and for the proof as answer to this question. For example: If students are asked to draw the three angle bisectors of a triangle on paper, many draw them intersecting in one point. They are thus not surprised to realize, while checking with the dynamic geometry software, that they were right, and do not feel a need for proof. However, when the same result is obtained as a by-product of a non-trivial investigation into the number of points of intersection of the angle bisectors of a quadrilateral, opportunities for surprise are created: The investigation of the quadrilateral's angle bisectors already offers many surprises; and then, when going over to the case of the triangle, students are surprised again because they do not expect the triangle's bisectors to intersect in a single point. (Dreyfus & Hadas, 1996).

b. Constructions under uncertainty conditions, where students try to construct, a figure satisfying given conditions. For an example of such an activity, see Hadas & Hershkowitz (1999).
c. Activities, in which the measurement and graphical options of the dynamic geometry software are used, in order to present the dynamic variations of a geometrical phenomenon in real time, also in the graphical and numerical modes. In such a context, questions and hypotheses raised in one mode may be answered in other modes, and in this way the investigating itself is enriched (Arcavi & Hadas, in press).
d. Activities in which one can not find any example for a conjecture one has made. Situations of uncertainty like this, lead to the dilemma mentioned above: Should one continue the empirical search or attempt to understand the impossibility? Hadas & Hershkowitz (1998) claim that, by careful design, based on experimentation and cognitive analysis of students' actions, situations can be constructed in which students will feel the need for proof. We conclude this section with an example of this type of activity, which illustrates how the dialectic process of the design of an activity with certain pedagogical purposes is integrated with cognitive research, as mentioned above (Stage II).

Are these 3 equal?

The development process of this activity had 3 cycles.
1. We started with a 'pre-research' version of the activity, which was tried out in a grade 9 classroom.

a. The side AC of a triangle is divided into 3 equal segments, by D and E. What can you say about the 3 angles created at the vertex B?



b. Construct the above as a dynamic figure on the computer. Investigate, by dragging and measuring, the relationships between the sizes of the 3 angles <ABD, <DBE, and <EBC. Explain!

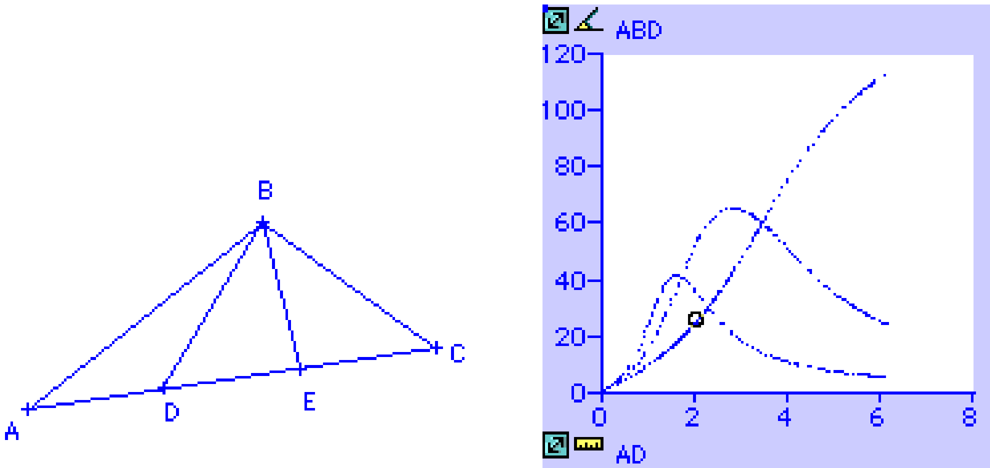
Most of the students guessed that the three angles were equal. After measuring, all students hypothesized that there are some cases in which the angles are equal. They struggled to find such a case and failed to explain why they could not. At last, the explanation was formulated in a whole class discussion.

2. In order to help students with the explanation, we designed and added two preliminary tasks.

Task 1.	Which values can the base angle of an isosceles triangle have? Explain!
Task 2a.	Investigate the median and the angle-bisector from the same vertex, in a 'dynamic triangle'. What can you say about the triangle when both segments coincide?
Task 2b.	Draw the median and the angle-bisector from the other two vertices. Try to find a situation in which two pairs coincide and the third pair does not. Explain!

This version formed the basis for a first semi-structured interview with two grade 9 students. They were not satisfied by visual considerations while dragging the vertices and changing the triangle, and therefore searched for a deductive explanation, based on the preliminary tasks (Hadas & Hershkowitz, 1998).

4. We decided to add an additional investigating tool -- the graphical representation of the varying angles as a function of 1/3 AC, for given AB and angle A (see the following figure).
5. This version, with the graph tool available to the students, was tried in a second interview with two other students. After matching the intersection points on the graph with the corresponding geometrical situations, the students started looking for an explanation why the angles must be different, rather than



searching for an example when they are equal. They ended up with a deductive reasoning (ibid.). In a third interview, after checking the three graphs for situations with equal angles, a pair of students tried to make the three intersection points as close as they could. They then realized that this *cannot* resolve their uncertainty, and that only the understanding *why* the three angles will never be equal, has the power to convince.

In the final version of the activity, which is now in the student materials for the entire population, the graph is suggested as an option to students who insist on finding an example with equal angles.

Activities of the 4 types mentioned above were developed in the second stage and served as the basis for the third stage, in which the entire geometry curriculum was shaped. In this curriculum the construction of concept images is rich and dynamic, and the need for proof emerges in comprehensive inquiry activities.

What does a Spreadsheet Contribute to Beginning Algebra?

Several experiments to include computers at the beginning algebra level were carried out during the last decade (Kieran, 1992). Heid (1995) describes the present and the expected shifts in a computer–intensive algebra as follows.

> “What was once the inviolable domain of paper-and-pencil manipulative algebra is now within easy reach of school level computing technology. This technology demands new visions of school algebra that shift the emphasis away from symbolic manipulation toward conceptual understanding, symbol sense, and mathematical modeling.

> No longer can the main purpose of algebra be the fine-tuning of techniques for by-hand symbolic manipulation or the acquisition of a predefined set of procedures for solving a fixed set of problems. Lesson after lesson of 'simplify these expressions' or 'solve these equations' will no longer characterize the school algebra experience. Students will spend far less time on many of these techniques, will execute a majority of them with computing technology, and will completely forgo the study of others. Although some of the attention now paid to symbolic representations will be rededicated to developing 'symbol sense', most class time will be spent in helping students develop a sense for how algebra can be used to explain the world around them. Applications of algebra will no longer be synonymous with 'age', 'coin', 'mixture', and 'distance-rate-time' word problems. Students will leave their school algebra experience with answers to such questions as 'What good is algebra?'" (p. 1)

These trends served as background for the development of a computer-based beginning algebra course, as part of the CompuMath project.

Pre-design considerations

The design of this algebra course was preceded, and hence influenced, by the design of the course on functions (aimed for Grade 9, see Stages II and III) and a course in statistical data analysis (aimed for Grade 7, see the section on statistics). As a result, the algebra course naturally adopted the basic characteristics of these earlier courses – such as basing most of the process of the construction of student knowledge on the investigation of problem situations, conducted in an environment of active peer interaction. However, in addition to the earlier developed basic assumptions, we had to consider the particularities and needs of the domain of beginning algebra, and of the students learning it.

An intuitive functional approach. The decision to base student activities on complex "real-life" or mathematical situations led us to investigations of processes of quantitative variations – such as measures of geometrical shapes, series expressed in a numerical or geometrical form, variation of weight, price, savings, distance etc. We also learned from other algebra curriculum projects (Heid et al, 1990; Yerushalmi, 1997) that the dynamic aspect of technological tools provides additional relevance to this approach. Our intention was to keep the formal aspect of the concept of function (definition, notation, mappings etc.) at a minimal level and to require students to investigate variation, as expressed by numerical series, algebraic expressions and graphs (see discussion in Stage I above).

A gradual and smooth transition between arithmetic and algebra. Traditionally, the transition from numbers to algebraic expressions is made in a sudden and arbitrary manner, causing a "didactical cut" and consequently many cognitive difficulties (Sutherland & Rojano, 1993; Ainley, 1996). Our intention was to allow students to make this transition at a slower pace – i.e., promoting a gradual and meaningful introduction of algebra, in parallel to the numerical representation of the investigated models.

Emphasis on generalizations and justifications. Generalizing is one of the central activities in algebra and is traditionally interpreted as expressing patterns, structures or processes symbolically. The traditional approach considered the translation of routine word problems into symbolic equations or expressions to be the main manifestation of this thinking skill, whereas our intention was to focus on the generalization of variations and patterns. Justifying belonged traditionally to the domain of geometry and was seriously neglected in algebra. Justifications are hardly needed as long as the main concern is translation to a symbolical representation and the manipulation of symbols – both activities being often disconnected from any context or meaning. We surmised that the investigation of meaningful situations and the use of computers would increase considerably the role and significance of justification in algebra.

Previous research on beginning algebra students (Friedlander et al., 1989) provided a notion about stages in generalizing a pattern (see the following figure) – stages that could be implemented in our case as a guiding scheme for the structure of an algebraic activity, involving processes of generalization and justification. This scheme includes transitions from the investigation of particular cases to generalizations, then to the justification of the generalized pattern and later on, to its implementation in additional cases.

<table>
<tr><td colspan="6">Launch
The students are presented with specific examples or specific examples are produced by them</td></tr>
<tr><td colspan="6">Towards a working generalization</td></tr>
<tr><td colspan="2">Producing additional examples</td><td colspan="2">Producing and solving examples with large numbers</td><td colspan="2">Solving ‘reversal’ tasks</td></tr>
<tr><td colspan="6">Towards an explicit generalization</td></tr>
<tr><td colspan="3">Verbal description of the observed pattern</td><td colspan="3">Symbolic description of the observed pattern</td></tr>
<tr><td colspan="6">Towards a justification</td></tr>
</table>

Choice of a technological tool. The following considerations, based on the criteria discussed in Stage I, led us to choosing spreadsheets as the technological tool for this course:

- Studies of students working with spreadsheets on arithmetic or beginning algebra problems (Sutherland & Rojano, 1993; Ainley, 1996) describe interesting and powerful thinking and strategies, evolving from students’ use of spreadsheets as a problem solving tool (*communicative power*).
- Spreadsheets seem to provide satisfactory answers to most of the project’s general requirements and in particular to those relevant to the domain of beginning algebra, such as *mathematization.*
- Spreadsheets (*Excel*) were successfully used as a technological tool for other area in mathematics (see the next section on statistics), as well as in science, thus satisfying the *generality* criterion.

<u>The Initial Development Stage</u>

Observations in the experimental classes in stage II produced many encouraging results. We will describe some of the more relevant ones:

- high student satisfaction and motivation to work (usually in pairs) on the designed activities;
- a considerable extension of the mathematical concepts, as compared to the traditional beginning algebra curriculum (e.g., early and natural introduction of algebraic variables, expressions and recursion formulae, an introduction to exponential and quadratic variation, arithmetical and geometrical sequences) – most of them presented, of course, informally;

- a considerable extension of the range and level of students' mathematical activities – such as *algebraic modeling*, which became the basis for any activity, *monitoring and justifying* the results produced by the developed, *mathematical discourse*, conducted during peer interaction, teacher interventions and class discussions,
- no significant technical difficulties in students' handling of the spreadsheets' algebraic aspects – such as the syntax of formulae or basic spreadsheet operations,
- an evolution of spreadsheet elements in the mathematical discourse – such as discussing the spreadsheet operations or discussing the solution of a problem, both orally and in writing, in an "*Excel* language" -- i.e., using *Excel* notation as a means of communication.

Moreover, the team located the following general issues related to the use of *Excel* as a mediator in the process of learning algebra:

1. *Generalization by recursion versus generalization by position* or *Spreadsheet formulae versus algebraic expressions*. Frequently, a number sequence can be obtained on a spreadsheet by either using position numbers, or by relating recursively to the previous number in the sequence. From a mathematical point of view, expressions that use the position as a variable reflect the underlying relation in a global way, whereas a recursive formula usually emphasizes a local aspect of the same relationship. *Excel* allowed students to combine the use of recursive formulae and dragging and thus to overcome the local characteristic of recursion. As a result, most students used recursion whenever possible. In some cases, like exponential growth, this may have been the only way available to them. However, the use of recursion does not allow one to find data beyond the ones included in the numerical table. We observed students extending their table to thousands of rows in order to answer a question that could have been solved by using a simple position formula. Thus, one of the issues at this stage was how to 'promote' position formulae, whenever they are mathematically more rewarding. An *abundance of numerical data* may also lead to an increased cognitive load, or to lack of motivation to monitor the obtained results.

   Similarly, there is an obvious equivalence between generalizing a pattern as a spreadsheet formula or as a standard algebraic expression. However, the difference between the two is sometimes more than syntactic. For example, when we required students to give an algebraic expression for the multiples of 5, with x specified as representing the position of each multiple in the sequence, some students made a direct transfer from the *Excel* recursion formula B2 + 5 (with the variable representing the previous number in the sequence) and produced the algebraic expression x + 5, rather than 5x, as expected.

   In summary, the spreadsheet's ability to produce large quantities of data by simple 'dragging' of formulae provides an excellent illustration of the meaning of a variable, an algebraic expression and the pattern of a variation. On the other hand the same ability can be 'abused', by preferring the extension of the numerical table to the use of a more mathematically sound strategy – such as constructing and solving an equation.
2. *Software transparency*. Students' willingness to monitor solution methods and their results was increased considerably by being released from computations and algebraic manipulations, by being able to relate to the meanings attached to the problem situations, and by being in a situation to discuss and argue about their ideas with their peers. On the other hand, in some cases we detected cognitive and technical difficulties in students' monitoring their solution processes and results. We related these difficulties not only to the cognitive load created by the abundance of numerical data but also, and more importantly, to the spreadsheets' lack of transparency – i. e. the 'disappearance' of the formulas that produced the numbers. The *Seal Designs* activity illustrates this issue:

Excerpt from the *Seal Designs* Activity.

All questions refer to 2×2 arrays of numbers such as the following:

3	**9**
5	**11**

Square 1

7	**13**
9	**15**

Square 2

10	**16**
12	**18**

Square 3

In a spreadsheet, construct a 'seal design' of formulae that can produce number squares like the ones given above

- First, enter a number of your choice in the upper left corner of the square.
- Then, use the name of this upper left cell to write formulae (and not numbers!), which will produce corresponding numbers in the other three cells of the square.
  Check whether your seal design produces the correct number squares.
- Enter the numbers 3, 7 and 10 in the upper left cell of your "seal" and check whether you obtain Squares 1, 2 and 3 given above.
  Investigate number squares of this kind.
- Find as many interesting properties as you can.
- Justify your findings – try to convince other people that the properties that you found are true for **all** the number squares of this kind.

The activity required the generalization of a 2×2 array of numbers, which can be characterized by the expressions x, x+6, x+2 and x+8. Then the students were asked to discover "interesting" patterns, common to all number squares of this kind. Two students, R and Y used correct formulae to produce the number arrays. They also found that the difference of the products of the two diagonals always equals 12. However, when they were asked to justify their claim (i. e., to show that $(x+2) \times (x+8)$ exceeded $x \times (x+8)$ by 12), they were distracted by the numbers shown on the spreadsheet for a particular array (7, 13, 9, 15) and attempted to compare $(x+2) \times 13$ and $x \times 15$ -- a mixture of remembered formulae and numbers taken from the specific square.

3. *Documentation of computer work*. The energy and motivation invested in computer work frequently led to incomplete documentation or a complete lack of any paper record of the computer work and its results.

In addition, we were led to consider the question of the *extent of instructions*. Many curriculum developers attempt to allow the students freedom in choosing their own solution path, but at the same time, they feel the need to include in the design of the task some guidance, which will help the students to complete the task successfully and to achieve its mathematical agenda. In spreadsheet-based activities, we had to decide on each occasion whether to recommend a certain structure for the numerical table (for example, its headings), to hint at or 'give away' the required formulae (especially at the beginning of the course) or to specify a sequence of spreadsheet operations needed to achieve a certain representation.

<u>Expansion</u>

Here, we will consider new emphases introduced as a result of the findings in the previous stage of development: hypotheses, graphical representations and generalization.

Hypotheses. We found that, in any process of inquiry, hypotheses were crucial in order to create meaningful learning situations and student involvement. Since students did not usually hypothesize spontaneously, we introduced specific requirements to predict at two stages:

- As part of getting acquainted with the problem situation, students were asked to predict some quantitative aspects of the outcome, *before* using the computer to obtain a table or make any other systematic attempt to solve the problem.
- Students were asked to predict some qualitative aspects (usually a rough sketch) of a graph, after they obtained a numerical table, but *before* they used the computer to produce the corresponding graph.

The *Savings* activity illustrates this issue. It is based on weekly doubling of an initially very small sum of money -- i.e., on exponential growth.

The 'Savings' problem situation.
Efrat's savings grow as follows:
At the end of the first week, she had 2 agorot (that is 0.02 shekel).
Each week, Efrat's savings grow by the amount that she has already saved up to then.

Students were required to make predictions at two stages:

a. First, the students were asked to compare the (exponential) savings of Efrat with the (linear) savings of other children, analyzed in a previous activity. They also had to estimate the amount of her savings by the end of one year. These predictions were made *before* using spreadsheets.
b. At a second stage, the students were required to use their numerical data on Efrat, to hypothesize and sketch the shape of the corresponding graph.

We found that predictions considerably increase students' willingness to monitor the output produced by the computer, and to analyze their solution if the outcome did not correspond to their prediction.

Graphical representations. Due to the influence of traditional beginning algebra, we tried to avoid graphical representations at the initial stage of development. However, our findings from the first stage of development showed that the use of graphical representations is a vital need in the developed activities, that there were no particular technical difficulties in students' handling of spreadsheet graphs and that the construction of graphs constituted a natural use of *Excel*'s abilities. As a result, we introduced graphs systematically, as one of four possible representations of data, models or solutions (verbal, numerical, algebraic and graphical). Metacognitive discussions of advantages and disadvantages of these representations were also conducted, both orally and in writing (for example, as journal items).

Discussing generalization methods. As mentioned above, we were quite aware of the importance of generalizations as one of the central processes in learning algebra. We were less aware, however, of students' tendency (especially when working with spreadsheets) to generalize recursively, rather than using the independent variable. In most cases, the second method seemed preferable to us, since the inclusion of the independent variable shows more clearly the underlying pattern. Therefore, right from the beginning of the course both, tasks and classroom discussions started to raise the issue of various ways of generalizing. Students were required to describe verbally or algebraically their generalizations in one or more specified method. For example, students were asked to construct on a spreadsheet a sequence, by using the numbers in the column of the position index, or to express in words the weekly balance of a person's savings by using the number of weeks (and *not* the previous balance) as variable.

We also found that the method of generalization was frequently influenced by the following characteristics of the task:

1 *Nature of variation*. At the stage of beginning algebra, linear models are employed most frequently. As mentioned before, spreadsheets tend to encourage recursive generalizations of linear relations. On the other hand, the same recursive methods allow students to analyze many non-linear models which could not be otherwise approached at this stage of learning algebra (see the *Savings* activity).
2 *Question design*. We found that presenting the first consecutive numbers, quantities or shapes of a sequence or a variation attracts recursion, whereas the presentation of one or two *non-consecutive* representatives of a similar variation, tends to encourage generalization using of the independent variable.
3 *Style of presentation*. A visual presentation of a variation or sequence (for example, sequences of dots or cubes, arranged in growing similar constructions) gives a strong meaning to general formulae, as a reflection of the counting method employed by the solver. Therefore, this presentation tends to encourage generalization by the position number – especially if non-consecutive representatives are used, as above.

In summary, while the spreadsheet may induce some characteristic obstacles to be avoided, we found that its language, its representational options and its mathematical capabilities combined into a powerful tool to support the establishment of the connection between arithmetic and algebra.

Statistics: Computerized Representations as Rhetorical Tools

Statistics is becoming ever more pervasive. Political, social, economic and scientific decisions are made on the basis of data. Statistical reports affecting virtually all aspects of our lives appear regularly in all the news media. Therefore, statistical literacy is becoming a major goal of the school curriculum. Gal (2000) suggests that statistical literacy is

> "people's ability to interpret and critically evaluate statistical information and data-based arguments appearing in diverse media channels; and to discuss their opinions regarding such statistical information."

We argue that the teaching/learning of statistics offers a particularly powerful testing-ground for probing fundamental questions regarding the role of computerized environments in curriculum development, as well as the links between the relationship between syllabus and curriculum.

More than other domains in school mathematics, the syllabus in school statistics is the object of intensive debate: to re-decide which statistics should be included in the school syllabus (Lajoie, 1998), which technology is appropriate for educational purposes (Ben-Zvi, 2000), and which type of research initiatives are needed. These issues led statistics educators (Graham, 1987; Hawkins et al, 1992; Garfield, 1995; Shaughnessy et al., 1996) to define *Exploratory Data Analysis* (EDA), or *Data Handling* as the content and framework of statistical education in schools. EDA is the discipline of organizing, describing, representing, and analyzing data, with a heavy reliance on visual displays as analytical tools and, in many cases, technology for making sense of data. EDA activities are often schematized by the slogans - looking at the data (preliminary analysis), looking between the data (comparisons), looking beyond the data (informal inference) and looking behind the data (context) (Curcio, 1989; Shaughnessy et al., 1996).

Following the recommendations to introduce stochastic (statistics and probability) concepts for all students from early stages (e.g., NCTM, 1989), new EDA instructional materials for elementary and secondary schools have been developed in many countries. In these curricula there is growing emphasis on graphical approaches, on students gathering their own data and carrying out investigations, on misuses and distortions and on probability simulations to generate data. We describe here a junior high school EDA course for grade 7 (age 13) developed as part of the CompuMath project.

In Israel, the official junior high school mathematics syllabus assigns 15 hours in grade 7 to cover basic statistics topics, and an additional 10 hours in grade 8 to introduce basic concepts of probability. The CompuMath EDA course was an extension of the previous curriculum development cycle, sharing with it basic approaches and the goal to design and create a learning environment in which students are engaged in meaningful mathematics. But, at the same time, it intended to make full use of the power of up-to-date technological tools to redesign and reshape the EDA learning environment.

<u>Stage I: Focus on the choice of a computer tool for EDA instruction</u>

At this stage, the curriculum development team systematically reviewed innovative statistics curricula, research literature and technological tools. We subsequently chose the statistical software to be used in class, and (re-)evaluated our educational goals and instructional strategies. We focus here on the choice of technological tool.

The types of software generally used in statistics instruction are manifold and include statistical packages, microworlds, tutorials, resources (including Internet resources), and teacher's meta-tools (Biehler, 1993, 1997; Ben-Zvi, 2000). *Statistical packages* include software for computing statistics and constructing visual representations of data, often based on a spreadsheet format to enter and store data. *Microworlds* consist of software programs to demonstrate statistical concepts and methods, including interactive experiments, exploratory visualizations, and simulations. Students can conceptualize statistics by manipulating graphs, parameters, and methods. For example, they allow the investigation of the effects of changing data on graphical representation, the effects of manipulating the shape of a distribution on its numerical summaries, or the effects of changing sample size on the distribution of the sample means. *Prob Sim* (Konold, 1995) and *Sampling Distributions* (delMas et al, 1998) are good examples of computer simulation microworlds.

Tutorials include programs developed to teach or tutor students on specific statistical skills, or to test their knowledge of these skills. The tutorial program is designed to take over parts of the role of the teacher and textbook, by supplying demonstrations and explanations, setting tasks for the students, analyzing and evaluating student responses, and providing feedback. The tutorials are often too dominant to leave enough room for students to construct knowledge autonomously. Examples include *ActivStats* (Currall et al, 1997), and *ConStatS* (Brewer, 1999). *Resources* consist of various resources to support teaching statistics including Internet resources. The development of the World Wide Web has produced unprecedented global means for teachers to easily share their ideas on ways to improve the teaching of statistics (Lock, 1998). *Teachers' meta-tools* create an interface that enables teachers to adopt software to their specific audience and educational goals. The categories listed above are not necessarily distinct, and in many cases specific software falls in more than one category.

We considered the above possible types against the CompuMath criteria for choosing technological tools (generality, mathematization, and communicative power), and opted for a *statistical package* and various *resources* (including Internet resources) for the EDA course. The specific educationally modified statistical package we chose at the pre-design stage was *Stats!* (LOGAL® Software, Inc.). *Stats!* is a data analysis program intended for middle and high school students in introductory statistics courses, stressing the analysis of real data using

EDA techniques. Students start with unordered data, typing both quantitative and qualitative data into any cell in a spreadsheet-like data table. Next, they can use the classification tool to order their data; and use tally sheets to display data by count or relative frequency. Graphic representations include pie charts, pictograms, bar charts, scatterplots, and accumulated frequency graphs. The representations can be enriched by also displaying the mode, mean, median, quartile, and boxplots. Students can manipulate the data interactively directly on the graphic representations, and compare two variables or populations on one display.

We hypothesized that such a manipulative power would foster mathematization and that the potential of *Stats!* in simultaneous displays of representation gave it communicative power. Finally, we chose *Stats!* because it was simple and suitable for all students, and did not demand adjustment for classroom activity. Also, it was still under development and the members of the curriculum development team were invited to function as advisors to the software programmers to include what we considered as educationally desirable procedures and functions.

The pre-design stage ended when we finished planning the main topics ("big ideas") of the curriculum (described in the next section); studied the features, advantages and limitations of the chosen software; and searched for suitable investigative situations (including real data). We were then ready to write the first version of some activities.

Stage II - The Initial Design

The second stage consisted mainly of a first design of activities and their implementation in a few grade 7 classrooms, accompanied by classroom research and student interviews, one of whose aims was to learn about their statistical intuitive conceptions and prior knowledge. These interviews showed a surprisingly broad knowledge of basic statistical concepts, such as averages and charts and their applications.

One major concern at this stage was the degree of openness of the activities. Our initial inclination, when planning a 'virtual activity', was to engage students in the investigation of data in a given context, and then to give them the freedom to choose research questions, tools and strategies to analyze the data. Thus, the 'virtual activity' was very open, instructions being minimal. The actual design of activities was however realized as an ongoing compromise, by trying to find the appropriate blend between open and closed.

Further, we experimentally added to the EDA course an extended activity – *a final project*, in which each pair of students identifies a problem and the questions they wish to investigate, suggest hypotheses, design the investigation, collect and analyze data, interpret the results, draw conclusions and present their main results to the class. Some of the topics students have chosen to investigate were: superstitions among students, attendance at football games, student ability and the use of the Internet, students' birth month, formal education of students' parents and grandparents, and road accidents in Israel. At this stage, we had little knowledge about the implementation of projects in class; for instance, how and when to introduce the project to students, how to guide their work or how to assess it.

In order to improve the learning materials and software, we observed and videotaped classrooms and students' work, interviewed students before and after the experimental implementation, read and analyzed student notebooks and final projects; and observed teachers in in-service workshops. All of these helped us to redesign the learning materials (see description below), to make them more attuned to student interests, motivation, capacities and interactions, and to calibrate the appropriate uses of technology (Ben-Zvi & Friedlander, 1997). For example, we changed investigation contexts, gave more attention to the "entry point" of the investigations, and improved the quality of the databases. Further, we faced two major problems in this stage: (a) shaping the relations between the classroom activities and the final project; and (b) the change of software. They are presented in turn.

(a) The relationship between the classroom activities and the final project

One option is to structure the components linearly; i.e. to locate the final project after the classroom activities, based on the theoretical assumption that students have to acquire the necessary statistical skills, tools and concepts, *before* they are able to begin on a large-scale final project of their own. A different option is to intertwine work on the final project with the classroom activities, assuming that the mutual effects are beneficial to learning. The hypothesis here is that the construction of statistical understanding is a complex non-linear process, that benefits from the combination of the semi-structured classroom activities and the self-propelled and open-ended final project. For this option, one has to choose carefully a starting point for the project work.

Specific classroom circumstances and the teacher's preferences also influenced the exact starting point of the final project. As expected, the classroom activities supplied some of the basic statistical approaches, concepts and skills needed for the project design. However, it became apparent that the early introduction of the final project into the course, motivated the students to take responsibility for their work and methods of inquiry, and gave them a sense of relevance, enthusiasm, and ownership. Students evaluated and applied new concepts and methods that were introduced in the classroom activities, not only in the given context of the different activities, but also in their own project. Thus, the project work gave them an added opportunity to experiment with the new concepts and methods, and often raised new statistical issues to explore, which were not originally part of the curriculum. The early start also provided more time for the project work. For example, some students spent several weeks to exploring, choosing and re-choosing an interesting and rich investigation topic. On the other hand, the early start of the final project caused it to dominate students' interest to a certain extent, which required special attention and flexibility on the teacher's part.

One of the most striking effects of the cross-fertilization between the project and the classroom activities is that the various representations of data, which were introduced in the activities as *didactic means* to convey statistical ideas to students, turned into *means of expression* by which the students presented their points of view, or tried to convince opponents, mainly during work on the projects. The students thus realized that the *representations could serve rhetorical functions*. This crucial point was taken advantage of by the curriculum development team in the third stage (see below).

(b) Changing the software

During the experimental implementation of the curriculum we were forced to reexamine the type of software we used. Although the software helped students to develop expertise in using data to solve real problems, it was limited in many respects and uncommon in schools. It had been created specifically for use in schools and consisted of a limited number of statistical procedures and representations. It was not *general* enough, and not mature enough, as a software. Therefore, we replaced *Stats!* by a *Spreadsheet* package (*Excel*). Our main reasons for this step were:

1. Spreadsheets are *common and familiar*. Spreadsheets, especially *Excel*, are now recognized as a fundamental part of computer literacy (Hunt, 1995). They are used in many areas of everyday life, as well as in other domains of mathematics curricula, and are available in many school computer labs. Hence, learning statistics with a spreadsheet helps to reinforce the idea that this is something connected to the real world. Moreover,

prior and in parallel to the learning of the EDA course, CompuMath students study algebra with *Excel* as described above.
2. Spreadsheets provide *direct access,* that allows students to view and explore data in different forms, investigate different models that may fit the data, manipulate a line to fit a scatter plot, etc.
3. Spreadsheets are *flexible and dynamic* allowing students to experiment with and alter displays of data. For instance, change, delete or add data entries in a table and consider the graphical effect of the change, or manipulate directly data points on the graph and observe the effects on a line of fit. Further, they are *adaptable;* namely, they provide students and teachers with control over the content and style of the output.

Unlike other topics described above, this stage with its field experiments and research, resulted in more than the design of a sequence of isolated activities. Most of the EDA activities were planned for 4-6 lessons, and the EDA course eventually consisted mainly of these extended activities and the final project.

<u>Stage III: Research</u>

Three directions of expansion were discussed in Stage III. Here, we focus on the relations between the constitution of a continuum of activities and learning trajectories, which is the object of intensive research (Ben-Zvi & Arcavi, 1998; Ben-Zvi, 1999; Ben-Zvi & Arcavi, submitted). We present (a) an example of the relation between research on learning and curriculum development (Men's' 100 Meter Race), and (b) an example of research on students' rhetorical use of representations (The Work Dispute).

(a) Men's' 100 Meter Race – Constructing meaning for trends

This study arose from a classroom activity. Students were presented with a spreadsheet table of the Olympic 100 meter records, the years in which they occurred (from 1896, the first modern Olympiad, to 1996), the athletes' names and country, etc. Their first task was to work in pairs to describe the data graphically and verbally, use the spreadsheet to produce a graph and discern trends. Our observations in several experimental classes showed that the students were able to engage quickly and with relative ease with the task. They were able to read the table of results, compare the records of consecutive Olympiads, consider the issue of outliers, sort the data, consider various graphs (some inappropriate for the given data), and create a time plot with a spreadsheet.

However, one frequent difficulty drew our attention: Students found it difficult to see *trends* in the given data (prior to graphing), to discern a pattern from the graph, and to report on it. Our interpretation was that at this stage cognitive issues relating to the connections between learning different topics appeared. Specifically, the relation between the EDA course and the algebra course, which is also based on the use of *Excel;* the deterministic nature of algebraic formulae interfered with the non-deterministic and disorganized nature of statistical data.

When we redesigned the activity, we included manipulation of scatter-graphs and a serious engagement with the notion of *trend*. The students were asked to use the spreadsheet to manipulate data graphs; i.e., change scales, delete an outlier, and connect points by lines, and to consider the effect of these changes on the shape of the graph. The objective was to prepare for a *design* task, in which they were asked to design a graph to support claims, such as:

1. Over the years, the times of the Olympic 100 meters race improved considerably.
2. Over the years, the changes in the times of the Olympic 100 meters race were insignificant.

3. Between 1948 and 1956, the times of the Olympic 100 meters race worsened considerably.

When manipulating data in one representation with immediate feedback in another, the computer provided the means to push the activity to a conceptual level, just as in geometry, algebra and functions (see Schwarz & Dreyfus, 1995).

Our observations indicate that the students constructed meanings through making connections between the investigation context, the data, and the graph. The computer assisted them in switching their discourse between the context, graph, and data, and thus helped them to construct meanings. Since students were found to be sensitive to this utilization of representations, we made many of our initial activities more rhetorical. For example, we modified the Olympic records activity as follows:

> "Two sports journalists argue about record times in the 100 meters. One of them claims that there seems to be no limit to human ability to improve the record. The other argues that at some time there will be a record, which will never be broken. To support their positions, both journalists use graphs."

Similarly, students were given a second database of the performances of Olympic women winners in the 100-meter race. The students were to draw a graph supporting the statement of a 'feminist activist' according to which *"women will sometime overcome men"* and another graph supporting the claim of a 'male chauvinist' according to which *"women will never run as fast as men"*.

This change in approach is characterized by the intention to change the status of representations from descriptive entities to entities that are to be judiciously constructed in order to attain a goal. The manipulation of data representations can yield graphs supporting any of the four arguments. Thus, in the third stage, activities were designed to engage students in *statistical literacy*. These activities resemble activities on functions and geometry, in which the students learn to discern between representatives (the material displays) and the meaning of these representatives. However, these activities were still, to some extent, of a prescriptive character. In the final example, the design of the activity leads students to use the technological tools to fulfill their own goals in an argumentative activity.

(b) The Work Dispute

This activity concerns workers in a printing company: The workers are in dispute with the management, who has agreed to a total increase of 10 percent in the salary bill. The dispute is about how this increase is to be divided among the employees. The students are given the present salary list of the one hundred employees, and an instruction booklet to guide them in their work. They are also provided with information about the national average and minimum salaries, Internet sites to look for data on salaries, newspaper articles about work disputes and strikes, and a reading list of background material. In the first part of the activity, groups of students are required to take a position in the dispute, and to clarify their arguments. Then, using the computer, they have to describe the distribution of salaries and use appropriate measures (median, mean, mode and range) to support their position. They learn about the effects of grouping data and the different uses of statistical measures in arguing their case. In the third part, their task is to suggest changes to the salary structure which satisfy the 10 percent constraint. They produce their proposal to solve the dispute, and design representations to support their position and refute opposing arguments. Finally the class meets for a general debate and votes for the winning proposal. The time spent on this activity is about seven class periods.

This activity led students to *manipulate representatives and data* for rhetorical use. They not only displayed averages and distributions of data, but designed the distribution of salaries to attain a desirable goal. This constitutes a jump comparable to between *learning to speak* a language and *speaking to learn*.

Taking a stand also made students check their methods, arguments and conclusions with extreme care. Criticism and counter-arguments by peers and teacher were a natural part of the activity. When the results of their work were not in line with their position, students were forced to persevere and search for more evidence and convincing arguments. Finally, after much refining the groups formulated their proposal.

In sum, we claim that statistics learning is a particularly powerful testing-ground for probing fundamental questions regarding the role of computerized environments in curriculum development, as well as the links between syllabus and curriculum. In the EDA course, students realized that representations and representatives could serve *rhetorical functions*, similar to their function in the work of statisticians, who select data, procedures, tools, and representations that support their perspective. This crucial point was taken advantage of by our team to extend the scope of the course beyond the learning of statistical concepts, to involve students in 'doing' statistics in a realistic context, through a set of semi-structured activities and an autonomous final project in a computerized environment.

Epilogue

In this chapter we have attempted to draw a picture of the compound activity of curriculum development for computer rich learning environments in mathematics. The account was organized along a few main dimensions of the curriculum development activity: Goals, participants, the potential of the technological tools, contents and approaches, the design of the materials to be used in the classrooms, research as an integral part of curriculum development, and implementation on a large scale. We tried to show how these dimensions change dynamically during the three main stages of a curriculum's development: pre-design considerations, initial design of isolated activities, and expansion.

The dynamically developing nature of these dimensions reaches beyond the stages of development and accompanies the project team throughout the life of the curriculum; some of the future-oriented dilemmas the decision makers and the project team face are the following.

- Technological tools develop at an exceedingly quick rate. A curriculum like CompuMath needs to take such development into account at two levels. At the level of more powerful and more user-friendly new versions of the same software packages, the adaptation needs to be done in order to allow schools to use all existing versions of the software. At a more profound level, educationally powerful new types of software may be expected to appear. This is particularly true for the algebra course. While we believe that we have chosen the most appropriate software available today for the functions, statistics and geometry courses, we are not yet fully satisfied with our approach to beginning algebra.
- Changes in the research dimension of the project emanate from two directions: Issues that arise from the actual development and implementation of the curriculum in classrooms may lead to research questions; and developments in the scientific discipline of mathematics education may entice us to investigate particular issues within our curriculum. In our approach, research is a necessary and integral part of curriculum development. It enables the development team to redesign a virtual learning activity into one for real students and classrooms in such a way that the intended change will happen. Such research follows the learners and investigates their learning in their own environment. As such it has socio-cultural characteristics. It also fits very well both, our

curriculum based interest in the role of collaborative problem solving with computers, and the current trend toward a socio-cultural outlook in the field of mathematics education. Ten years ago, cognitive issues had a much larger weight in our curriculum based research. And in the near future, the question of combining the cognitive and the socio-cultural approaches is likely to take on added importance, specifically the question where in shared knowledge the individual knowledge is hidden (Hershkowitz, 1999).

- While the above dimensions are more or less in the hands of the project team, others demonstrate the power of decision-makers outside the project team on the future development and realization of the project. For example, in an ideal environment the students can be autonomous and use technological tools whenever they feel the need for it. For this purpose, computers need to be available in each classroom on a permanent basis and in each student's home. In today's typical CompuMath environment, computers are in a lab and students may use the lab about once per week. The resources and organization to overcome these limitations of the learning environment depend on powers beyond the curriculum team.
- Similarly, as discussed in Stage III, implementation on a large scale is a vast task. Problems arise because teachers, students and the teaching-learning processes are in a sense 'unknown': they have a much lower degree of interaction with the team members than the stage II participants. It seems reasonable to create an electronic system for communication with teachers and students. In the CompuMath project we piloted this idea in one mid-sized town with some success. But the bulk of work on such a 'curriculum maintenance system' is waiting to be carried out, and requires enormous resources.
- Finally, the most serious compromise that the CompuMath developers had to make was to teach the contents of the given mathematics syllabus, which was formulated three decades ago and does not take the potential of computerized tools into consideration. In Stage I we discussed the gap between syllabus and curriculum, and how it was bridged by creating suitable standards for mathematics learning and teaching, and by careful design. A large number of mathematical problem situations were created and connected into a web of meanings of which the syllabus constitutes but the barest outline. Moreover, in some instances, the syllabus was more of a liability than an asset. The time has come to reconsider the contents of the syllabus itself. For instance, the mere existence of Computer Algebra Systems raises questions concerning the role of algebraic manipulations such as solving equations: To what extent do students have to reach mastery in solving linear or quadratic equations, when they have a tool at their disposal which solves compound equations including the linear and the quadratic ones? What kind of mathematical knowledge and insight may students gain using such a tool? In short, the time has come to reconsider the syllabus in the light of the technological tools' potential in mathematics learning.Curriculum development projects are not limited in time and scope. Any project is necessarily based on previous projects as well as on conditions imposed from the outside. And a project's success can eventually be measured only by the influence it exerts beyond the immediate classrooms in which it is implemented and beyond the few years during which it is taught.

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